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On the Kinetic Approach to Many-Body Problems in General Relativity

P. YODZIS†

Department of Physics, New Mexico State University, Las Cruces, New Mexico

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Abstract

A new formulation of general relativistic kinetic theory, which includes the effects of 'collisions' in a self-consistent way, is given. It is found that if collision terms are necessary in the kinetic equation, then Einstein's field equations must be modified as well.

Recently, there has been quite some interest in the kinetic approach to many-body problems in general relativity (Chernikov, 1962; Lindquist, 1966; Hakim, 1968; Ehlers *et al.*, 1968; Ipser & Thorne, 1968; Bel, 1969; Ipser, 1969a, b). The usual method of deriving the kinetic equation, however, obscures an important feature of general relativistic kinetic theory; namely that if collision terms are necessary in the kinetic equation, then Einstein's field equations must be modified as well.

We shall present here a new derivation which reveals this curious phenomenon (due essentially to the non-linearity of the Einstein equations), and which perhaps also gives some insight on the physics of the gravitational plasma. Our method is a straightforward generalization of the Klimontovich (1967) approach to plasma kinetic theory.

Consider, then, a system of N particles,‡ each of mass m, which interact only gravitationally. We assume that 'hard collisions' are negligible, that is that the four-velocity of each particle is a continuous (four-vector-valued) function of the particle's proper time. The number N of particles is understood to be large.

Let the position and four-velocity of the Ath particle, $A = 1, \ldots, N$, be denoted by $x_A{}^\mu(s_A)$ and $u_A{}^\mu(s_A)$, where s_A is the proper time measured along the world line of the Ath particle.§ Each particle moves along a geodesic:

$$\frac{du_A{}^\xi(s_A)}{ds_A} = -u_A{}^\mu(s_A)\, u_A{}^\nu(s_A)\, \Gamma^\xi_{\mu\nu}[x(s_A)] \tag{1}$$

† Address after 1 October 1970: School of Theoretical Physics, Dublin Institute for Advanced Studies, Dublin 2.

‡ These 'particles' may, of course, be stars or even galaxies.

§ Greek indices run from 1 to 4, and we shall sometimes denote the partial derivative with respect to x^μ by ∂_μ. Our space-time metric has signature +2.

where $\Gamma^{\xi}_{\mu\nu}$ are the Christoffel symbols corresponding to the collective gravitational field of the remaining particles.

Define the *microscopic* or *fine-grained* distribution function $\mathscr{F}(x,u)$ by

$$\mathscr{F}(x,u) \equiv \int_{-\infty}^{\infty} ds_1 \ldots ds_N \, \mathscr{G}(x,u;s) \tag{2}$$

where

$$\mathscr{G}(x,u;s) \equiv \frac{1}{-g(x)} \sum_{A=1}^{N} \delta[x - x_A(s_A)]\, \delta[u - u_A(s_A)]$$

with $g(x)$ the determinant of the metric tensor $g_{\mu\nu}(x)$. The δ's are four-dimensional Dirac delta-distributions; thus we are treating the particles as 'point particles'. One could conceivably incorporate into the formalism finite particles with internal structure, but this seems a bit ambitious at present.

It is convenient to treat $\mathscr{F}$ as a distribution function in eight-dimensional phase space although, of course, it is non-zero only for u which obey $u^{\mu} u_{\mu} = -1$. Call a connected hypersurface in phase space *spacelike* if its projection into configuration space is a spacelike hypersurface; that is, if it consists of a four-dimensional volume in u-space and a spacelike hypersurface in space-time. Clearly, each orbit in the congruence of particle orbits in phase space crosses a spacelike hypersurface in phase space at most once. An invariant measure on such a hypersurface is $d\sigma_{\mu} du$, where $d\sigma_{\mu}$ is an invariant space-time hypersurface measure and

$$du \equiv d^4 u \sqrt{-g} \equiv du^1\, du^2\, du^3\, du^4 \sqrt{-g}$$

The meaning of $\mathscr{F}$ is that the integral over any spacelike hypersurface in phase space of the quantity $u^{\mu}\mathscr{F}$ is equal to the number of particle orbits which cross that hypersurface. If we integrate over all u-space, we get the *numerical flux vector*†

$$N^{\mu}(x) \equiv \int du\, u^{\mu} \mathscr{F}(x,u)$$

The energy-momentum tensor for the system of particles, which is just a sum of 'matter' energy-momentum tensors, is given by

$$T^{\mu\nu}(x) = m \int du\, u^{\mu} u^{\nu} \mathscr{F}(x,u) \tag{3}$$

One can show, by straightforward calculation [using (1)], that

$$\sum_{A} \frac{\partial}{\partial s_A} \mathscr{G}(x,u;s) = -\left(u^{\mu} \frac{\partial}{\partial x^{\mu}} - u^{\mu} u^{\nu} \Gamma^{\xi\nu}_{\mu}(x) \frac{\partial}{\partial u^{\xi}}\right) \mathscr{G}(x,u;s)$$

† See, for instance, p. 21 of Synge (1957).

Hence, from equation (2),

$$\left(u^\mu \frac{\partial}{\partial x^\mu} - u^\mu u^\nu \Gamma^\xi_{\mu\nu}(x) \frac{\partial}{\partial u^\xi}\right) \mathscr{F}(x, u) = 0 \tag{4}$$

This is an exact kinetic equation for the microscopic distribution function $\mathscr{F}$. In fact, it is just an identity.

The kinetic equation (4), along with Einstein's field equations

$$G^{\mu\nu} = -\kappa T^{\mu\nu} \tag{5}$$

with $T^{\mu\nu}$ given by (3), provide an exact description of the system of point particles. As they stand, however, these equations are not very useful, for $\mathscr{F}$, $T^{\mu\nu}$ and $g_{\mu\nu}$ are all very complicated, randomly fluctuating, entities. To get a tractable theory, we must smooth out these quantities by some sort of averaging process.

We shall use an ensemble average, which we denote by $\langle\ \rangle$. Let

$$g_{\mu\nu} \equiv \langle g_{\mu\nu} \rangle + \delta g_{\mu\nu}$$

The quantities $\delta g_{\mu\nu}$ are *fluctuations*. (Note that, by definition, the ensemble average of a fluctuation is zero). Because the ensemble average does not involve the variables x^μ, u^μ, it preserves tensorial type. In particular, $\langle g_{\mu\nu} \rangle$ are the covariant components of a second-rank tensor, the kernel of which we shall denote by $\langle g \rangle$:

$$\langle g \rangle_{\mu\nu} \equiv \langle g_{\mu\nu} \rangle$$

The tensor $\langle g \rangle$ is the average space-time metric; its contravariant components are defined by

$$\langle g \rangle_{\mu\nu} \langle g \rangle^{\nu\lambda} = \delta_\mu{}^\lambda$$

Notice that although $\langle g^{\mu\nu} \rangle$ are the contravariant components of a second-rank tensor, $\langle g^{\mu\nu} \rangle \neq \langle g \rangle^{\mu\nu}$, for

$$\delta_\mu{}^\lambda = g_{\mu\nu} g^{\nu\lambda} = (\langle g_{\mu\nu} \rangle + \delta g_{\mu\nu})(\langle g^{\nu\lambda} \rangle + \delta g^{\nu\lambda})$$

and taking the ensemble average of both sides of this equation yields

$$\delta_\mu{}^\lambda = \langle g \rangle_{\mu\nu} \langle g^{\nu\lambda} \rangle + \langle \delta g_{\mu\nu} \delta g^{\nu\lambda} \rangle \tag{6}$$

The ensemble average of any product of fluctuations is called a *central moment*; these central moments are intimately related to the correlation functions.† We shall set

$$\langle g^{\mu\nu} \rangle \equiv \langle g \rangle^{\mu\nu} + h^{\mu\nu}$$

Notice that if the central moments occurring in equation (6) vanish, then so does h.

† As outlined on pp. 57–59 of Klimontovich (1967).

For a treatment of fluctuations, it is convenient to define the new distribution function

$$f \equiv \sqrt{(-g)}\mathscr{F}$$

The kinetic equation (4) then reads

$$\left(u^\mu \frac{\partial}{\partial x^\mu} - u^\mu u^\nu \Gamma^\xi_{\mu\nu} \frac{\partial}{\partial u^\xi} - u^\mu \Gamma^{\xi\nu}_{\mu\nu}\right) f = 0, \tag{7}$$

and the energy-momentum tensor (3) is

$$T^{\mu\nu} = m \int d^4 u \sqrt{(-g)}\, u^\mu u^\nu \mathscr{F} = m \int d^4 u\, u^\mu u^\nu f \tag{8}$$

If $a_{\mu\nu}$ are the covariant components of any second-rank tensor, define the functional

$$\Gamma_{\mu\nu,\,\xi}\{a\} \equiv \partial_\mu a_{\nu\xi} + \partial_\nu a_{\mu\xi} - \partial_\xi a_{\mu\nu}$$

Defining the fluctuation δf of f by

$$f = \langle f \rangle + \delta f$$

we can write equation (7) as

$$\left[u^\mu \frac{\partial}{\partial x^\mu} - u^\mu u^\nu \Gamma_{\mu\nu,\,\xi}\{\langle g\rangle + \delta g\}(\langle g^{\xi\eta}\rangle + \delta g^{\xi\eta}) \frac{\partial}{\partial u^\eta} - \right.$$
$$\left. - u^\mu \Gamma_{\mu\nu,\,\xi}\{\langle g\rangle + \delta g\}(\langle g^{\xi\nu}\rangle + \delta g^{\xi\nu})\right][\langle f \rangle + \delta f] = 0$$

Ensemble averaging this equation, we obtain

$$\left[u^\mu \frac{\partial}{\partial x^\mu} - u^\mu u^\nu \Gamma_{\mu\nu,\xi}\{\langle g\rangle\}\langle g\rangle^{\xi\eta} \frac{\partial}{\partial u^\eta} - u^\mu \Gamma_{\mu\nu,\,\xi}\{\langle g\rangle\}\langle g\rangle^{\xi\nu}\right]\langle f \rangle$$
$$= u^\mu u^\nu A_{\mu\nu} + u^\mu A_\mu \tag{9}$$

where

$$A_{\mu\nu} \equiv \Gamma_{\mu\nu,\,\xi}\{\langle g\rangle\} h^{\xi\eta} \frac{\partial}{\partial u^\eta}\langle f \rangle + \Gamma_{\mu\nu,\,\xi}\{\langle g\rangle\} \frac{\partial}{\partial u^\eta}\langle \delta g^{\xi\eta} \delta f \rangle +$$
$$+ \langle g^{\xi\eta}\rangle \frac{\partial}{\partial u^\eta}\langle \Gamma_{\mu\nu,\,\xi}\{\delta g\}\, \delta f \rangle + \langle \Gamma_{\mu\nu,\,\xi}\{\delta g\}\, \delta g^{\xi\eta}\rangle \frac{\partial}{\partial u^\eta}\langle f \rangle +$$
$$+ \frac{\partial}{\partial u^\eta}\langle \Gamma_{\mu\nu,\,\xi}\{\delta g\}\, \delta g^{\xi\eta}\, \delta f \rangle$$

and A_μ is a similar expression involving central moments.

Performing a similar analysis of the Einstein equations (5), with $T^{\mu\nu}$ given by (8), one obtains

$$\overset{\langle\,\rangle}{G}{}^{\mu\nu} + \Delta G^{\mu\nu} = -\kappa m \int d^4 u\, u^\mu u^\nu \langle f \rangle \tag{10}$$

where the symbol $\langle\,\rangle$ over any functional of the metric tensor and its derivatives means that the functional is to be evaluated for the metric $\langle g \rangle$, and $\Delta G^{\mu\nu}$ is a complicated expression involving central moments, which we shall not write out explicitly. It is the necessary modification of the Einstein equations alluded to earlier.

Suppose the system of particles is sufficiently diffuse that correlations are negligible (see below). Then the central moments are negligible, and (9) and (10) reduce to

$$\left(u^\mu \frac{\partial}{\partial x^\mu} - u^\mu u^\nu \overset{\langle\rangle}{\Gamma}{}^\xi_{\mu\nu} \frac{\partial}{\partial u^\xi} - u^\mu \overset{\langle\rangle}{\Gamma}{}^\nu_{\mu\nu}\right) \langle f \rangle = 0 \tag{11}$$

$$\overset{\langle\rangle}{G}{}^{\mu\nu} = -\kappa m \int d^4 u u^\mu u^\nu \langle f \rangle \tag{12}$$

Equation (11) is the general relativistic Vlasov equation.† If one identifies $(-\langle g \rangle)^{-1/2} \langle f \rangle$ with the 'macroscopically' defined distribution function of other authors, equations (11) and (12) are precisely the equations which some of these authors have applied to diffuse gravitational plasmas.

Correlations will be important for a system in which there are many close encounters, that is, configurations in which the behavior of some small set of particles depends more on the parameters of those particles than on the collective gravitational field of the plasma as a whole. Such a situation is characterized by large deflections of some particles over a relatively short proper path length.

Clearly for the universe in its present state, regarded as a gravitational plasma of galaxies, correlations are negligible. In earlier states of the universe, though, correlations probably played an important role. In individual galaxies, treated as stellar gravitational plasmas, correlations are probably largely negligible,‡ but in questions of galactic cosmogony correlations may be significant. Certainly, if one wishes to apply this theory to the structure of individual stars, one cannot neglect correlations.

Let us, then, reconsider equations (9) and (10), this time for a system in which there are many close encounters; the central moments (that is, $A_{\mu\nu}$, A_μ and $\Delta G^{\mu\nu}$) now cannot be neglected. One can, as some authors have done, replace the right-hand side of (9) by a phenomenological collision term. But it is not then consistent to assume that $\Delta G^{\mu\nu}$ is negligible, and phenomenological reasoning gives no clue as to what it should be.

A consistent approach would be to generate [as in electromagnetic plasma kinetic theory (Klimontovich, 1967)] from equations (5) and (7) an infinite set of coupled equations for the central moments of all orders.§ For an electromagnetic plasma, one can usually justify neglecting in these equations all but the lowest-order central moments, which reduces the infinite set of equations to a finite (even small) set. Because of the absence in a

† It is usually referred to in the literature as the '(collisionless) Boltzmann equation', but from our point of view the term 'Vlasov equation' is more appropriate.

‡ See, for instance, the estimate of ter Haar (1969), which is based upon Newtonian gravitation.

§ The order of a central moment is its order as a formal multinomial in the fluctuations.

gravitational plasma of Debye screening, one must proceed in this case with some circumspection. Indeed, with N equal to 10 or 24, and in the Newtonian limit, solutions obtained by numerical integration (van Aldaba, 1968) indicate that multiple close encounters tend to be more important than binary ones. This would indicate that the higher-order central moments cannot be neglected. However, as remarked by ter Haar (1969), it is not at all clear to what extent one can infer the behavior of large N systems from the behavior of small N systems.

This rather difficult problem deserves thorough study, for, as indicated above, correlations are likely to play an important role in many interesting physical problems, and consistent solutions of these problems will require some knowledge of the correlation term $\Delta G^{\mu\nu}$.

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Erratum

Vol. 3, No. 3 (1970), pp. 205–231. Darryl Leiters' paper was received 15 November 1969 and not 1970 as stated.